21-241 – Solution to Homework assignment week #4

Laurent Dietrich
Carnegie Mellon University, Fall 2016, Sec. F and G

1 Exercises

1. Let a, b be any real numbers and

$$(S)\begin{cases} x + \ \ y = a \\ 2x - 3y = b \end{cases}$$

Solve (S). Rewrite (S) using the column-by-column approach. Interpret the results.

2. In the following, determine whether W is a subspace of V.

   i) $V = \mathbb{R}^3, W = \left\{ \begin{bmatrix} x \\ y \\ x+y+1 \end{bmatrix} \middle| \; x, y \in \mathbb{R} \right\}$

   ii) $V = \mathcal{M}_{nn}(\mathbb{R}), W$ is the subset of *diagonal matrices*, that is, matrices with non-zero entries only on the diagonal (the one from top-left to bottom-right).

3. Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be linear and such that

$$T\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} \text{ and } T\begin{bmatrix} 1 \\ -3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$$

Can you compute $T\begin{bmatrix} a \\ b \end{bmatrix}$ for any $a, b \in \mathbb{R}$ (justify your answer) ? If yes, do so.

4. Find two matrices A and B such that $(A+B)^2 \neq A^2 + 2AB + B^2$. When is this relation actually satisfied ?

5. Prove Theorem 1)v) from Chapter 5. That is, if A is any $m \times n$ matrix then

$$I_m A = A = A I_n$$

2 Solution

1. One can reduce system ($R_2 \leftarrow R_2 - 2R_1$) and see that it has only one solution

$$x = \frac{3a+b}{5} \qquad y = \frac{2a-b}{5}$$

Using the column-by-column approach, (S) reads

$$x\begin{bmatrix}1\\2\end{bmatrix} + y\begin{bmatrix}1\\-3\end{bmatrix} = \begin{bmatrix}a\\b\end{bmatrix}$$

We just saw that this problem has a unique solution no matter what a and b are : this means that $\begin{bmatrix}1\\2\end{bmatrix}$ and $\begin{bmatrix}1\\-3\end{bmatrix}$ span $\mathbb{R}^2$. [1]

2. i) W is not a subspace since the 0-vector is not in it.

   ii) W is a subspace : just observe that the zero-matrix is diagonal and that the sum of two diagonal matrices as well as any scalar multiple of a diagonal matrix stay diagonal.[2]

3. We saw in Exercise 1 that any $\begin{bmatrix}a\\b\end{bmatrix}$ can be computed as a (unique) linear combination of $\begin{bmatrix}1\\2\end{bmatrix}$ and $\begin{bmatrix}1\\-3\end{bmatrix}$. By linearity, we can then compute any $T\begin{bmatrix}a\\b\end{bmatrix}$. We get

$$T\begin{bmatrix}a\\b\end{bmatrix} = T\left(\frac{3a+b}{5}\begin{bmatrix}1\\2\end{bmatrix} + \frac{2a-b}{5}\begin{bmatrix}1\\-3\end{bmatrix}\right) = \frac{3a+b}{5}\begin{bmatrix}3\\1\\-1\end{bmatrix} + \frac{2a-b}{5}\begin{bmatrix}1\\0\\2\end{bmatrix}$$

4. We can use

$$A = \begin{bmatrix}1 & 0\\0 & 0\end{bmatrix} \qquad B = \begin{bmatrix}0 & 1\\0 & 0\end{bmatrix}$$

Observe that

$$(A+B)^2 = \begin{bmatrix}1 & 1\\0 & 0\end{bmatrix} \qquad A^2 + 2AB + B^2 = \begin{bmatrix}1 & 2\\0 & 0\end{bmatrix}$$

We can actually compute

$$(A+B)^2 = A^2 + AB + BA + B^2$$

[1] And even more : not only they span it but there is a *unique* linear combination for each vector. We say that these two vectors form a *basis* of $\mathbb{R}^2$ (see later in class).

[2] Write it down : outside the diagonal one gets only $[A+B]_{ij} = a_{ij} + b_{ij} = 0 + 0 = 0$ if $i \neq j$ and $[\lambda A]_{ij} = \lambda a_{ij} = \lambda \times 0 = 0$ if $i \neq j$

so the equality holds if and only if $AB + BA = 2AB$, i.e. iff

$$AB = BA$$

That is, when A and B commute. More generally, the binomial theorem holds for matrices if and only if they commute.

5. Let A be a $m \times n$ matrix and let $1 \leq i \leq m, 1 \leq j \leq n$. We can compute the product $I_m A$ which is a $m \times n$ matrix as well and we have

$$(I_m A)_{ij} = \sum_{k=1}^{m} (I_m)_{ik} A_{kj}$$

But observe that $(I_m)_{ik} = 1$ if and only if $i = k$ and is $= 0$ otherwise. Thus, only the term $k = i$ remains in the sum, so

$$(I_m A)_{ij} = A_{ij}$$

This is valid for all $1 \leq i \leq m, 1 \leq j \leq n$ so this means that $I_m A = A$.

The proof for AI_n is completely similar.