21-268 → section

+24

HW #8

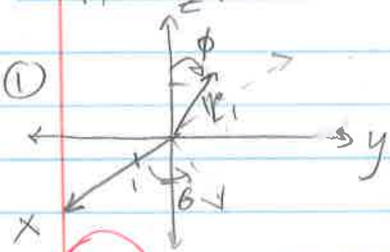
$x = \rho \sin\phi \cos\theta$

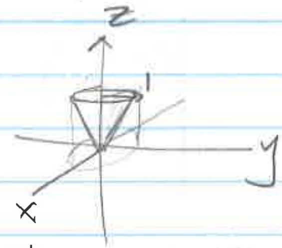
$y = \rho \sin\phi \sin\theta$

$z = \rho \cos\phi$

$dV_{xyz} = \rho^2 \sin\phi \; dV_{\rho\phi\theta}$

(1)





BOUNDS ON z ⇒ cone

x6

BOUNDS ON y: circle of radius 1 ⇒ $\phi \to 0, \pi/4$

⇒ $\theta \to 0, 2\pi$

Bounds on x: (-1, 1) → Full circle

$\rho \to$ goes from $0 \to \frac{1}{\cos\phi}$

$x^2+y^2+z^2 = \rho^2$

$\rho\cos\phi = 1$

$\to \rho = \frac{1}{\cos\theta}$

$$\to \int_0^{\pi/4}\int_0^{2\pi}\int_0^{\frac{1}{\cos\phi}} \rho^2 \cdot \rho^2 \sin\phi \; d\rho \, d\theta \, d\phi$$

$$\to \int_0^{\pi/4}\int_0^{2\pi} \frac{1}{5}\rho^5 \sin\phi \Big]_0^{\frac{1}{\cos\phi}} = \int_0^{\pi/4}\int_0^{2\pi} \frac{\sin\phi}{5\cos^5\phi} \; d\theta \, d\phi$$

$$= -2\pi \int_0^{\pi/4} \frac{-\sin\phi}{5\cos^5\phi} \, d\phi = -\frac{2\pi}{5}\left[-\frac{1}{4}(\cos^{-4}\phi)\right]_0^{\pi/4}$$

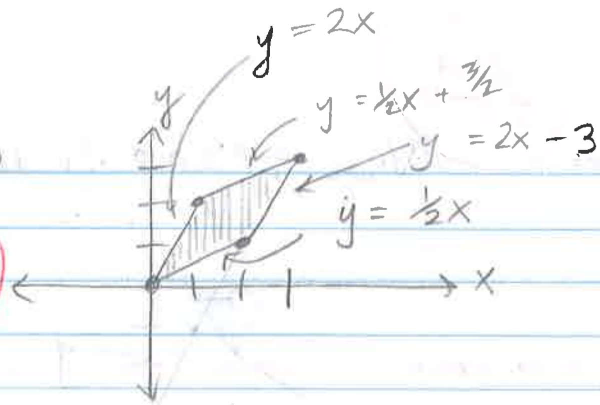
$$= \frac{2\pi}{20\cos^4(\pi/4)} - \frac{2\pi}{20\cos^4(0)}$$

$\frac{4}{16}$

$$= \frac{2\pi}{20\cdot\left(\frac{\sqrt{2}}{2}\right)^4} - \frac{2\pi}{20}$$

$$= \frac{4\pi}{10} - \frac{\pi}{10} = \boxed{\frac{3\pi}{10}}$$

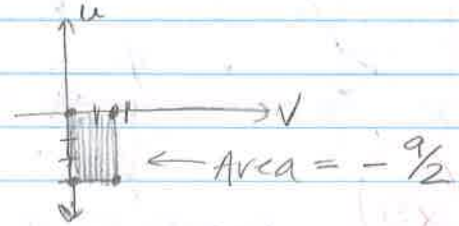
② +6



$\Rightarrow$ $y - 2x = u$

$y - \frac{1}{2}x = v$

$\Rightarrow$



$$\frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{\frac{\partial(u,v)}{\partial(x,y)}}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -\frac{2}{3} & \frac{2}{3} \\ -\frac{1}{3} & \frac{4}{3} \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}$$

$y = u + 2x$

$\Rightarrow u + 2x - \frac{1}{2}x = v$

$\Rightarrow x = (v-u)\frac{2}{3}$

$\Rightarrow y - \frac{1}{2}\left(\frac{2}{3}(v-u)\right) = v$

$\Rightarrow y = v + \frac{1}{3}v - \frac{1}{3}u$

$y = \frac{4}{3}v - \frac{1}{3}u$

$$dA_{xy} = \left(-\frac{8}{9} + \frac{2}{9}\right) dA_{uv} = -\frac{2}{3}\, dA_{uv}$$

$-\frac{2}{9} - \frac{8}{9}$

$$\Rightarrow \iint_{R_{uv}} \left(2\left(\tfrac{2}{3}v - \tfrac{2}{3}u\right)^2 + 5\left(\tfrac{2}{3}v - \tfrac{2}{3}u\right)\left(\tfrac{4}{3}v - \tfrac{1}{3}u\right) + 2\left(\tfrac{4}{3}v - \tfrac{1}{3}u\right)^2\right)\left|-\tfrac{2}{3}\right| dA_{uv}$$

$$\Rightarrow \iint_{R_{uv}} 2\left(\tfrac{4}{9}v^2 - \tfrac{8}{9}vu + \tfrac{4}{9}u^2\right) - 5\left(\tfrac{8}{9}v^2 - \tfrac{10}{9} + \tfrac{2}{9}u^2\right) + 2\left(\tfrac{16}{9}v^2 - \tfrac{8}{9}vu + \tfrac{1}{9}u^2\right)$$

$$\left[\tfrac{8}{9}v^2 - \tfrac{40}{9}v^2 + \tfrac{32}{9}v^2\right] \quad \left[-\tfrac{16}{9}vu + \tfrac{50}{9}vu - \tfrac{16}{9}vu\right] \quad \left[\tfrac{8}{9} - \tfrac{10}{9} + \tfrac{2}{9}\right]$$

$$\Rightarrow \iint_{R_{uv}} (2uv)\left|-\tfrac{2}{3}\right| dA_{uv}$$

$$\Rightarrow \int_{-3}^{0}\int_{0}^{3/2} +\frac{4}{3}uv\, dv\, du = \int_{-3}^{0} \left. +\frac{4}{6}v^2u \right]_0^{3/2} du = \int_{-3}^{0} \frac{+36}{24}u\, du$$

$$\Rightarrow \left.\frac{36}{48}u^2\right]_{-3}^{0} = \frac{-36\cdot 9}{48} = \boxed{\frac{-27}{4}}$$

③ $x = uv \qquad y = u + v \qquad (u,v): 2 \le u \le 3$

$y = u + \frac{x}{u} \qquad\qquad\qquad\qquad 0 \le v \le 1$

x ⑩

$0 = u^2 - uy + x$

$$\boxed{u = \frac{y \pm \sqrt{y^2-4x}}{2}}$$

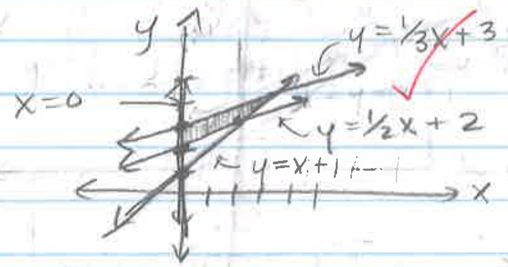
opposite signs

$v = y - u$

$v = \frac{y \mp \sqrt{y^2-4x}}{2}$ ← since v is always +, this one is +

since $2 \le u \le 3$, we need $u = \frac{y + \sqrt{y^2-4x}}{2}$ in order for 2 to be the lower bound.

$$u = \frac{y + \sqrt{y^2-4x}}{2} \; ✓, \qquad v = \frac{y - \sqrt{y^2-4x}}{2} \; ✓$$



THUS, since there's one solution for u & one for v, this transformation is 1-to-1.

$4 = y + \sqrt{y^2-4x}$, $6 = y + \sqrt{y^2-4x}$

$y = \frac{1}{2}x + 2$ $y = \frac{1}{3}x + 3$

$0 = y - \sqrt{y^2-4x}$, $2 = y - \sqrt{y^2-4x}$

$x = 0 \qquad y = x + 1$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} v & u \\ 1 & 1 \end{vmatrix} = |v-u| = u - v,$$

since $v < u$

$$\iint_{R_{xy}} y \, dA_{xy} = \iint_{R_{uv}} (u+v)(u-v)\, dA_{uv}$$

$$= \int_2^3 \int_0^1 (u^2 - v^2)\, dv\, du = \int_0^1 \left[\tfrac{1}{3}u^3 - v^2 u\right]_2^3 dv = \int_0^1 \frac{27}{3} - 3v^2 - \frac{8}{3} + 2v^2$$

$\frac{19}{3} - v^2$

$$= \int_0^1 \frac{19}{3} - v^2\, dv = \frac{19}{3}v - \frac{1}{3}v^3 \Big]_0^1 = \frac{19}{3} - \frac{1}{3} = \frac{18}{3}$$

$$= \boxed{6}$$

SCRATCHWORK

$\cancel{1+}\sqrt{y^2-4x} = y - 1 \qquad (4-y) = \sqrt{y^2-4x}$

$\cancel{y^2} - 4x = \cancel{y^2} - 4y + 4 \qquad y^2 - 4x = 16 - 8y + y^2 \qquad (6-y)^2 = y^2 - 4x$

$4y = 4 + 4x \qquad 8y = 16 + 4x \qquad 36 - 12y + \cancel{y^2} = \cancel{y^2} - 4x$

$y = x + 1 \qquad \boxed{y = \frac{1}{2}x + 2} \qquad 36 + 4x = 12y$

$y = \frac{1}{3}x + 3$

(4) $R_{xy} = \{(x,y) : 5x^2 + 2xy + 2y^2 \leq 1\}$

$x = u + v,\ y = -2u + v$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} \Rightarrow \boxed{dA_{xy} = 3\,dA_{uv}}$$

$$\Rightarrow \iint_{R_{uv}} \sqrt{5(u+v)^2 + \underbrace{2(u+v)(-2u+v)}_{2(-2u^2+uv-2uv+v^2)} + \underbrace{2(-2u+v)^2}_{(4u^2-4uv+v^2)}} \cdot 3 \ dA_{uv}$$

$$\Rightarrow \iint \sqrt{5u^2 + \cancel{10uv} + 5v^2 - 4u^2 - \cancel{2uv} + 2v^2 + 8u^2 - \cancel{8uv} + 2v^2} \cdot 3 \ dA_{uv}$$

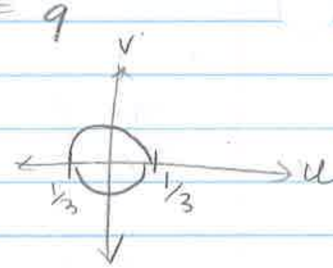
$$\Rightarrow \iint_{R_{uv}} \sqrt{9u^2 + 9v^2} \cdot 3 \ dA_{uv} = \iint_{R_{uv}} 9\sqrt{u^2+v^2}\ dA_{uv}$$

$5x^2 + 2xy + 2y^2 \leq 1$

$\Rightarrow 9u^2 + 9v^2 \leq 1 \Rightarrow u^2 + v^2 \leq \frac{1}{9}$

$\Rightarrow u \to -\frac{1}{3}, \frac{1}{3}$

$v \to -\sqrt{\frac{1}{9} - u^2},\ \sqrt{\frac{1}{9} - u^2}$



CHANGE to polar:

$v = r\sin\theta$

$u = r\cos\theta$

$$\int_0^{2\pi}\int_0^{1/3} 9r^2\, dr\, d\theta$$

$$\Rightarrow \int_0^{2\pi} 3r^3 \Big]_0^{1/3} d\theta = \frac{1}{9}\theta \Big]_0^{2\pi} = \boxed{\frac{2\pi}{9}}$$